

A rate of asymptotic regularity for the Mann iteration of κ -strict pseudo-contractions

Daniel Ivan^{1,*}, Laurențiu Leuştean^{2,3},

¹ 17 Lumbervale Ave, Toronto M6H 1C7, Ontario, Canada

² Faculty of Mathematics and Computer Science, University of Bucharest,
Academiei 14, P.O. Box 010014, Bucharest, Romania

³ Simion Stoilow Institute of Mathematics of the Romanian Academy,
P.O. Box 1-764, RO-014700, Bucharest, Romania

E-mails: idaniel.ivan@gmail.com, Laurentiu.Leustean@imar.ro

Abstract

In this paper we apply methods of proof mining to obtain a uniform effective rate of asymptotic regularity for the Mann iteration associated to κ -strict pseudo-contractions on convex subsets of Hilbert spaces.

MSC: 47J25; 47H09; 03F10.

Keywords: Proof mining; Effective bounds; Asymptotic regularity; Strict pseudo-contractions; Mann iteration

1 Introduction

Let H be a real Hilbert space, $C \subseteq X$ a nonempty closed convex subset, $T : C \rightarrow C$ be a mapping and $0 \leq \kappa < 1$. T is said to be a κ -strict pseudo-contraction if for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|x - Tx - (y - Ty)\|^2. \quad (1)$$

This class of nonlinear mappings was introduced in the 60's by Browder and Petryshyn [2]. Nonexpansive mappings coincide with 0-strict pseudo-contractions.

The Mann iteration [6, 8, 3] starting with $x \in C$ is defined by

$$x_0 := x, \quad x_{n+1} := (1 - \lambda_n)x_n + \lambda_nTx_n, \quad (2)$$

*The research of Daniel Ivan was done while he was a member of the project PN-II-ID-PCE-2011-3-0383 at Simion Stoilow Institute of Mathematics of the Romanian Academy.

where (λ_n) is a sequence in $(0, 1)$. By letting $\lambda_n = \lambda$ for all $n \in \mathbb{N}$, we get the Krasnoselskii iteration [6] as a special case. In the sequel, we consider a sequence (λ_n) satisfying the following conditions:

$$\kappa < \lambda_n < 1 \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=0}^{\infty} (\lambda_n - \kappa)(1 - \lambda_n) = \infty. \quad (3)$$

Assuming that T has fixed points and (λ_n) satisfies (3), Marino and Xu [9] proved the weak convergence of the Mann iteration (x_n) to a fixed point of T . Their result generalizes the one obtained by Browder and Petryshyn for the Krasnoselskii iteration. Furthermore, as an immediate consequence one gets Reich's result [10] for nonexpansive mappings in Hilbert spaces.

As it is the case with many results on the weak or strong convergence of non-linear iterations, the first step in their proofs consists in getting the *asymptotic regularity* of (x_n) , i.e. the fact that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ for all starting points $x \in C$. This is a very important property, introduced by Browder and Petryshyn [1] for the Picard iteration $x_n = T^n x$. The following result is implicit in [9]:

Theorem 1.1. *Let C be a convex subset of a Hilbert space H , $T : C \rightarrow C$ be a κ -strict pseudo-contraction such that T has fixed points. Assume that (λ_n) satisfies (3). Then $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ for all $x \in C$.*

In this paper we apply proof mining methods to obtain a finitary, quantitative version of a generalization of Theorem 1.1, computing a uniform rate of asymptotic regularity for the Mann iteration (x_n) , i.e. a rate of convergence of $(\|x_n - Tx_n\|)$ towards 0. The fact that we can get such a result is guaranteed by logical metatheorems for Hilbert spaces proved by Kohlenbach [5]. Moreover, as an immediate consequence of our main result, we obtain a quadratic rate of asymptotic regularity for the Krasnoselskii iteration.

2 Main result

Given $x \in X$ and $b, \delta > 0$, we use the notation

$$Fix_\delta(T, x, b) = \{y \in C \mid \|x - y\| \leq b \text{ and } \|y - Ty\| < \delta\}$$

and we say that T has *approximate fixed points* in a b -neighborhood of x if $Fix_\delta(T, x, b) \neq \emptyset$ for all $\delta > 0$. If T has a fixed point $p \in C$, then for all $x \in C$ and any $b \geq d(x, p)$, we have that $p \in Fix_\delta(T, x, b)$ for all $\delta > 0$.

Let us recall that a rate of divergence for a divergent series $\sum_{n=0}^{\infty} a_n$ is a mapping

$$\theta : \mathbb{N} \rightarrow \mathbb{N} \text{ satisfying } \sum_{k=0}^{\theta(n)} a_k \geq n \text{ for all } n \in \mathbb{N}.$$

The main result of this paper is the following finitary, quantitative version of a generalization of Theorem 1.1, where the hypothesis of T having fixed points is

weakened to the one that T has approximate fixed points in a b -neighborhood of x for some $x \in C$ and $b > 0$.

Theorem 2.1. *Let H be a Hilbert space, $C \subseteq H$ a nonempty convex subset and $T : C \rightarrow C$ be a κ -strict pseudo-contraction, where $0 \leq \kappa < 1$. Assume that (λ_n) is a sequence in $(\kappa, 1)$ satisfying $\sum_{n=0}^{\infty} (\lambda_n - \kappa)(1 - \lambda_n) = \infty$ with rate of divergence $\theta : \mathbb{N} \rightarrow \mathbb{N}$. Let $x \in C, b > 0$ be such that $\|x - Tx\| \leq b$ and T has approximate fixed points in a b -neighborhood of x . Then $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and*

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, b, \theta)(\|x_n - Tx_n\| < \varepsilon), \quad \text{where} \quad \Phi(\varepsilon, b, \theta) = \theta \left(\left\lceil \frac{b^2}{\varepsilon^2} \right\rceil \right). \quad (4)$$

Proof. See Section 4. □

Browder and Petryshyn proved [2] that if C is bounded, then T has fixed points. Hence, by letting b to be an upper bound for the diameter d_C of C , we get that $\text{Fix}_\delta(T, x, b) \neq \emptyset$ for all $x \in C$. As a consequence of our main theorem, for bounded C , the Mann iteration (x_n) is asymptotically regular with rate of asymptotic regularity Φ given by (4), where $b \geq d_C$.

Furthermore, if $\lambda_n = \lambda \in (\kappa, 1)$ then one can easily verify that

$$\theta(n) := \left\lceil \frac{1}{(\lambda - \kappa)(1 - \lambda)} \right\rceil n$$

is a rate of divergence for the series $\sum_{n=0}^{\infty} (\lambda - \kappa)(1 - \lambda) = \infty$. Hence, we get in this case that

$$\Phi(\varepsilon, b, \lambda, \kappa) = \left\lceil \frac{1}{(\lambda - \kappa)(1 - \lambda)} \right\rceil \left\lceil \frac{b^2}{\varepsilon^2} \right\rceil \quad (5)$$

is a quadratic in $1/\varepsilon$ rate of asymptotic regularity for the Krasnoselskii iteration (x_n) .

Since 0-strict pseudo-contractions coincide with nonexpansive mappings, our results generalize with slightly changed bounds the ones obtained by Kohlenbach [4] for the Mann iteration, and Browder and Petryshyn [2] for the Krasnoselskii iteration associated to nonexpansive mappings in Hilbert spaces. We point out that in [4], Kohlenbach computes, applying also proof mining methods, rates of asymptotic regularity for the Mann iteration of a nonexpansive mapping in the more general class of uniformly convex Banach spaces, generalized further by the second author to a class of uniformly convex geodesic spaces [7].

3 Some useful lemmas

In the sequel, H is a Hilbert space, $C \subseteq H$ is a nonempty convex subset and $T : C \rightarrow C$ is a κ -strict pseudo-contraction. Furthermore, (λ_n) is a sequence in $(0, 1)$ and (x_n) is the Mann iteration starting with $x \in C$, defined by (1). The following identities in Hilbert spaces will be used in the sequel.

Lemma 3.1. *Let $x, y \in H$ and $t \in [0, 1]$. Then*

$$\begin{aligned}\|x + y\|^2 &= \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle \text{ and } \|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle \\ \|tx + (1-t)y\|^2 &= t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2.\end{aligned}$$

Lemma 3.2. (i) *For all $y, z \in C$,*

$$\|Tz - y\|^2 \leq \|z - y\|^2 + \kappa\|z - Tz\|^2 + (\kappa + 1)\|y - Ty\|^2 + 2\|z - Ty\|\|y - Ty\|.$$

(ii) *For all $y \in C$ and all $n \geq 0$*

$$\begin{aligned}\|x_{n+1} - y\|^2 &\leq \|x_n - y\|^2 - (\lambda_n - \kappa)(1 - \lambda_n)\|x_n - Tx_n\|^2 \\ &\quad + 2\|y - Ty\|(\|x_n - y\| + 2\|y - Ty\|).\end{aligned}$$

Proof. (i)

$$\begin{aligned}\|Tz - y\|^2 &= \|(Tz - Ty) + (Ty - y)\|^2 \\ &= \|Tz - Ty\|^2 + \|Ty - y\|^2 + 2\langle Tz - Ty, Ty - y \rangle \\ &\leq \|z - y\|^2 + \kappa\|(z - Tz) - (y - Ty)\|^2 + \|Ty - y\|^2 \\ &\quad + 2\langle Tz - Ty, Ty - y \rangle \\ &= \|z - y\|^2 + \kappa\|z - Tz\|^2 + (\kappa + 1)\|y - Ty\|^2 - 2\langle z - Tz, y - Ty \rangle \\ &\quad + 2\langle Tz - Ty, Ty - y \rangle \\ &= \|z - y\|^2 + \kappa\|z - Tz\|^2 + (\kappa + 1)\|y - Ty\|^2 + 2\langle z - Ty, Ty - y \rangle \\ &\leq \|z - y\|^2 + \kappa\|z - Tz\|^2 + (\kappa + 1)\|y - Ty\|^2 + 2\|z - Ty\|\|y - Ty\|\end{aligned}$$

(ii)

$$\begin{aligned}\|x_{n+1} - y\|^2 &= \|\lambda_n x_n + (1 - \lambda_n)Tx_n - y\|^2 = \|\lambda_n(x_n - y) + (1 - \lambda_n)(Tx_n - y)\|^2 \\ &= \lambda_n\|x_n - y\|^2 + (1 - \lambda_n)\|Tx_n - y\|^2 - \lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2 \\ &\leq \lambda_n\|x_n - y\|^2 + (1 - \lambda_n)\|x_n - y\|^2 + (1 - \lambda_n)\kappa\|x_n - Tx_n\|^2 \\ &\quad + (1 - \lambda_n)(\kappa + 1)\|y - Ty\|^2 + 2(1 - \lambda_n)\|x_n - Ty\|\|y - Ty\| \\ &\quad - \lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2 \quad \text{by (i)} \\ &= \|x_n - y\|^2 - (\lambda_n - \kappa)(1 - \lambda_n)\|x_n - Tx_n\|^2 \\ &\quad + (1 - \lambda_n)(\kappa + 1)\|y - Ty\|^2 + 2(1 - \lambda_n)\|x_n - Ty\|\|y - Ty\| \\ &\leq \|x_n - y\|^2 - (\lambda_n - \kappa)(1 - \lambda_n)\|x_n - Tx_n\|^2 \\ &\quad + 2\|y - Ty\|(\|x_n - y\| + 2\|y - Ty\|),\end{aligned}$$

since $\|x_n - Ty\| \leq \|x_n - y\| + \|y - Ty\|$.

□

In particular, if p is a fixed point of T , then for all $n \geq 0$,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (\lambda_n - \kappa)(1 - \lambda_n)\|x_n - Tx_n\|^2. \quad (6)$$

A very important property of the Mann iteration is the following one

Lemma 3.3. [9] *The sequence $(\|x_n - Tx_n\|)$ is nonincreasing.*

Lemma 3.4. *Let $y \in C$ and $b \geq \max\{\|x - Tx\|, \|x - y\|\}$ and $c \geq \|y - Ty\|$. Then for all $n \geq 0$,*

- (i) $\|x_n - y\| \leq (n+1)b$ and $\|Tx_n - y\| \leq (n+2)b$.
- (ii) $\|x_{n+1} - y\|^2 \leq \|x_n - y\|^2 - (\lambda_n - \kappa)(1 - \lambda_n)\|x_n - Tx_n\|^2 + 2((n+1)b + 2c)\|y - Ty\|$.

Proof. (i) By induction on n , taking into account that, for all n , we have that $\|x_{n+1} - y\| \leq \lambda_n\|x_n - y\| + (1 - \lambda_n)\|Tx_n - y\|$ and that $\|x_n - Tx_n\| \leq \|x - Tx\| \leq b$, by Lemma 3.3.

(ii) Apply (i) and Lemma 3.2.(ii).

□

4 Proof of Theorem 2.1

Let us denote, for simplicity, $\Delta := \sum_{n=0}^{\Phi} (\lambda_n - \kappa)(1 - \lambda_n)\|x_n - Tx_n\|^2$.

Claim: $\Delta \leq b^2$.

Proof of claim: We prove that $\Delta \leq b^2 + \sigma$ for all $\sigma \in (0, 1)$. Apply the fact that T has approximate fixed points in a b -neighborhood of x , and we get for

$$\delta := \frac{\sigma}{(\Phi + 1)(\Phi b + 2b + 2)}$$

an $y \in C$ such that $\|x - y\| \leq b$ and $\|y - Ty\| < \delta < \frac{1}{2}$. We can apply Lemma 3.4.(ii) with b as in the hypothesis and $c := 1/2$ to obtain

$$\begin{aligned} \Delta &\leq \sum_{n=0}^{\Phi} (\|x_n - y\|^2 - \|x_{n+1} - y\|^2) + 2 \sum_{n=0}^{\Phi} ((n+1)b + 1)\|y - Ty\| \\ &= \|x_0 - y\|^2 - \|x_{\Phi+1} - y\|^2 + 2\|y - Ty\| \left(\frac{(\Phi + 1)(\Phi + 2)b}{2} + (\Phi + 1) \right) \\ &\leq \|x - y\|^2 + \|y - Ty\|(\Phi + 1)(\Phi b + 2b + 2) < b^2 + \sigma. \end{aligned}$$

The claim is proved.

Since $(\|x_n - Tx_n\|)$ is nonincreasing, it is enough to prove that there exists $N \leq \Phi$ such that $\|x_N - Tx_N\| \leq \varepsilon$. Assume by contradiction that for all $n = 0, \dots, \Phi$ one has $\|x_n - Tx_n\| > \varepsilon$. It follows that

$$\begin{aligned} \Delta &= \sum_{n=0}^{\Phi} (\lambda_n - \kappa)(1 - \lambda_n) \|x_n - Tx_n\|^2 > \sum_{n=0}^{\Phi} (\lambda_n - \kappa)(1 - \lambda_n) \varepsilon^2 \\ &= \varepsilon^2 \sum_{n=0}^{\theta(\lceil b^2/\varepsilon^2 \rceil)} (\lambda_n - \kappa)(1 - \lambda_n) \geq \left\lceil \frac{b^2}{\varepsilon^2} \right\rceil \cdot \varepsilon^2 = b^2. \end{aligned}$$

Thus, we have got a contradiction. \square

Acknowledgements:

This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI, project number PN-II-ID-PCE-2011-3-0383.

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